STABLE DIRECT ADAPTIVE PERIODIC CONTROL USING ONLY PLANT ORDER KNOWLEDGE

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Abstract

The main contribution of this paper is to put stability requirements for convergence of *direct* adaptive periodic controllers on equal footing with requirements for indirect adaptive periodic control, as set forth by Lozano [20]. The resulting stability condition is simply that the plant order is known a-priori. No other prior plant knowledge is used (e.g., relative degree, high-frequency gain, etc.), and persistent excitation is not required. More importantly, no assumption or knowledge is required as to whether the plant is minimum or nonminimum phase. A numerical example is given to demonstrate the method, and some guidelines are given for improving the adaptive transient response.

1 INTRODUCTION

An intriguing property associated with generalized sampling mechanisms is their ability to relocate transmission zeros of the plant. The potential benefit of sampling for zero relocation was noted in the paper by Astrom, Hagander and Sternby [3]. Subsequent research investigated applications of generalized sampling mechanisms to such problems as robust control, simultaneous stabilization, sensitivity minimization, and *zero* placement, cf., [11][12][18][19].

Generalized sampling can take many different forms, e.g., multirate sampling, periodic control, generalized sample-and-hold, etc. Most approaches have an interpretation as a mathematical "lifting" where a serial to parallel conversion is performed on the plant input and output signals, and mappings are considered between the vectorized quantities.

In Lozano [20] an important lifting was introduced for which the transmission zeros are located at the origin, Such liftings are denoted here as zero annihilation (ZA) liftings. General conditions characterizing the ZA property can be found in Bayard [6], along with several *extended horizon* lifting versions which satisfy the ZA conditions. Extended horizon liftings have the advantage of reducing required control torque and the size of the transient response, and have been applied to problems in optical instrument pointing [9], and structural vibration damping [8].

The transmission zeros of the ZA lifted plant are at the origin regardless of whether the original plant is minimum or nonminimum phase. This is important since it provides a means by which a nonminimum phase plant can be "transformed" into a minimum phase lifted plant. In light of this property, it is not surprising that several stable adaptive control approaches for nonminimum phase systems have been developed based on such liftings [5] [20] [22] [25].

Of particular interest are the adaptive controllers of Lozano [20] [21] [22]. These adaptive controllers are of the indirect type, i.e., the plant parameters are estimated first, and are then used to compute the control gains. A main result of Lozano is that only the plant order is required to be known to establish stability.

The present paper will consider *direct* adaptive control for the same class of liftings. The main contribution of this paper is to put stability requirements for convergence of direct adaptive periodic controllers on equal footing with requirements for indirect adaptive periodic control, i.e., that the plant order is known a-priori. No prior knowledge of the plant relative degree or high-frequency gain is used, and persistent excitation is not required. More importantly, no assumption or knowledge is required as to whether the plant is minimum or nonminimum phase.

2 BACKGROUND

The lifting of Lozano is briefly reviewed [20]. Consider the single-input single-output state-space realization,

$$x(t+1) = \bar{A}x(t) + \bar{b}u(t) \tag{1}$$

$$y(t) = \bar{c}^T x(t) \tag{2}$$

where $x \in R^n$ is the state vector, u and y are the scalar plant input and output, respectively, and $\bar{A}$, $\bar{b}$, $\bar{c}$ *are* system matrices of appropriate dimensions.

Then it is shown in Lozano [20], that the plant input and output in (1)(2) satisfy,

$$Y(t+2n) = AY(t) + H''U(t+n) + HU(t) + H'U(t-n) \tag{3}$$

where,

$$U^T(t) = [u(t), \ldots, u(t+n-1)] \tag{4}$$

$$Y^T(t+n) = [y(t), \ldots, y(t+n-1)] \tag{5}$$

$$H = \mathcal{O}\mathcal{C} \tag{6}$$

$$A = \mathcal{O}\bar{A}^{2n}\mathcal{O}^{-1} \tag{7}$$

$$H' = \mathcal{O}\bar{A}^{n}\mathcal{C} - \mathcal{O}\bar{A}^{2n}\mathcal{O}^{-1}G \tag{8}$$

$$H'' = \begin{bmatrix} 0 & & & \\ c^T b & & & \\ & & \ddots & \\ c^T A^{n-2} b & \cdots & c^T b & 0 \end{bmatrix} \tag{9}$$

and $\mathcal{O}$ is the system observability matrix and $\mathcal{C}$ is the system reachability matrix. Let U_k denote U at time $t = 2kn$, $k = 0, 1, \ldots,$ and enforce (by design) the constraint,

$$U(t-n) = 0 \quad for \quad t = 2kn \tag{10}$$

Furthermore, let Y_k denote $Y(t+2n)$ at time $t = 2kn, k = 0,1,\ldots$. This notation definea a lifting whose sampling structure is shown in Fig. 1. As seen in the figure, (10) forces the input to be zero every alternate window of length n. It is shown by Lozano that using (10), model (3) can be written as,

$$Y_k = AY_{k-1} + HU_k \tag{11}$$

As shown in Fig. 1, the output is controlled in alternate windows, which are staggered in time with respect to the nonzero input windows.

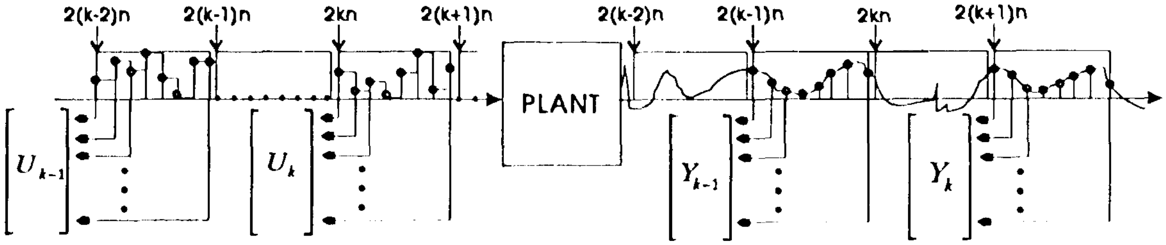

Figure 1: Mechanization of lifting.

It is emphasized that any controllable and observable linear tirne-invariant tplant (l)(2) can be lifted into the form (11) using only knowledge of its plant order [20]. Furthermore, the nonsingularity of the leading coefficient H is ensured simply by the reachability and observability of the original (unlifted) plant and does not depend on whether the true (unlifted) plant is minimum or nonminimum-phase.

The discussion will focus on developing a stable adaptive law for (1 1).

A rearrangement of (11) gives the equivalent plant representation,

Linear Control *Form*

$$U_k = KY_{k-1} + LY_k = \Theta r_k \tag{12}$$

where,

$$K = -H^{-1}A;\quad L = H^{-1} \tag{13}$$

$$\Theta = [K \mid L];\quad r_k = [Y_{k-1}^T \mid Y_k^T]^T \tag{14}$$

Representation (12) is said to be in Linear Control *Form* (cf., Goodwin and Sin [15]) since the input control is written as a linear function of observed signals, One important advantage of this parametrization is that a deadbeat controller can be written directly in terms of the gains K and L *as* follows,

Deadbeat Control

$$U_k^\circ = KY_{k-1} + LY_k^d = \Theta r_k^d \tag{15}$$

$$r_k^d = [Y_{k-1}^T \mid (Y_k^d)^T]^T \tag{16}$$

where Y_k^d is a specified trajectory to be tracked by Y_k. Hence, it is only necessary to estimate K and L in (12) and then "copy" the estimates for implementing the deadbeat control (15).

Lozano has developed several adaptive control approaches [20] [22] [21], based on the representation (11). Lozano's approaches are '(indirect" in the sense that the plant parameters A and H are first estimated from (11) and then used to compute the control gains K and L in (15) using the formulas in (13) (note that only K *is* required for adaptive regulation). From (13) it is seen that this requires a numerical inversion of the estimate of H each iteration. In order to ensure invertibility of this estimate, Lozano introduces a modification in [20] based on a polar decomposition.

In contrast to Lozano's approach, the present paper will focus on a "direct" adaptive scheme. In a direct scheme, the gains K and L in control law (15) are estimated directly from the plant representation (12). Earlier stable direct adaptation schemes have been developed for periodic control in Ortega [25] and Bayard [5]. The present direct adaptive approach is similar to those in [25][5], except that the Recursive Least Squares (RLS) algorithm will be used rather than simple normalized projection, and tuning will be breed on minimization of the input error rather than the output error. The advantage of this approach is that only *knowledge of the plant order is* required for stability, i.e., the requirements for prior partial Markov parameter information in Bayard [5] and Cauchy Index constraints in Ortega [25] have been relaxed.

An added benefit of direct adaptive control is that numerical inversion of the estimate H of H is avoided. However, even though $\hat{H}$ is not inverted, its nonsingularity is still required to ensure adaptive stability. Hence, the polar decomposition introduced by Lozano will be needed to complete the. stability proof.

Several simulation studies indicating the performance advantagea (i.e., bounds on transient response, convergence time, etc.), of using a direct adaptive approach with RLS adaptation can be found in Jakubowski et. al., [16][17]. However, these earlier studies were conducted without any mechanism to ensure stability, and several counterexamples to convergence presently exist. Compared to [16][17], the present paper introduces several modifications to ensure stability, and to provide a theoretical framework for using this clam of algorithms.

3 STABLE ADAPTIVE CONTROL

In this section, a stable direct adaptive controller is defined for the plant lifting (1 1).

3.1 Input Prediction Error

Given an estimate $\hat{\Theta}_{k-1}$ of@ available at time k, one can construct the input prediction,

$$U_k^p = \hat{\Theta}_{k-1} r_k \tag{17}$$

and the associated input prediction error,

$$E_k \triangleq U_k^p - U_k = \Phi_{k-1} r_k \tag{18}$$

where,

$$\Phi_{k-1} \triangleq \hat{\Theta}_{k-1} - \Theta \tag{19}$$

3,2 Normalized Signals

For adaptation purposes, it is useful to define the following normalized quantities,

$$\tilde{r}_k = \frac{r_k}{1+\eta_k}; \quad \tilde{r}_k^d = \frac{r_k^d}{1+\eta_k}; \quad \tilde{E}_k = \frac{E_k}{1+\eta_k} \tag{20}$$

where the normalization factor is defined by,

$$\eta_k = \gamma_k(||Y_k|| + ||Y_{k-1}||) \tag{21}$$

and upper and lower bounds are specified on γ_k,

$$\bar{\gamma} \geq \eta_k \geq \underline{\gamma} > 0 \tag{22}$$

Dividing through by $1+\eta_k$ in (18) defines the normalized prediction error equation,

$$\tilde{E}_k \triangleq \Phi_{k-1} \tilde{r}_k \tag{23}$$

3.3 Adaptation Algorithm

Equation (23) is a linear-in-the-parameter error expression for which many adaptation methods apply. The discussion here will focus on the Matrix Parameter Recursive Least Squares (MP-RLS) algorithm,

MP-RLS Adaptation Algorithm

$$\hat{\Theta}_k = \hat{\Theta}_{k-1} - \frac{\tilde{E}_k \tilde{r}_k^T F_{k-1}}{1 + \tilde{r}_k^T F_{k-1} \tilde{r}_k} \tag{24}$$

$$F_k = F_{k-1} - \frac{F_{k-1} \tilde{r}_k \tilde{r}_k^T F_{k-1}}{1 + \tilde{r}_k^T F_{k-1} \tilde{r}_k} \tag{25}$$

It is shown in Appendix B (see also [7]), that the MP-RLS algorithm satisfies the following properties,

1'1: $\Phi_k F_k^{-1} = \Phi_{k-1} F_{k-1}^{-1} = \dots = \Phi_0 F_0^{-1}$

P2: $v_k \le v_{k-1} \le \dots \le v_0$ where $v_k \triangleq tr\{\Phi_k F_k^{-1} \Phi_k^T\}$

P3: $\bar{\sigma}(F_k) \le \bar{\sigma}(F_{k-1}) \le \dots \le \bar{\sigma}(F_0)$

P4: $\lim_{k\to\infty} \tilde{E}_k = 0$

P5: $tr\{\Phi_k \Phi_k^T\} \le v_0 \cdot \bar{\sigma}(F_0)$

P6: $\lim_{k\to\infty} ||\hat{\Theta}_k - \hat{\Theta}_{k-1}||_f = 0$, where $||\cdot||_f$ is the Frobenious norm

P7: $\lim_{k\to\infty} F_k = F_\infty$

P8: $\lim_{k\to\infty} F_{k-1}\tilde{r}_k = 0$

1'9: $\lim_{k\to\infty} \hat{\Theta}_k = \Theta_\infty = \Theta + \Phi_0 F_0^{-1} F_\infty$

3.4 Adaptive Control Law - Discussion

An adaptive control law is defined by replacing Θ in (15) by its estimate, i.e.,

$$U_k = \hat{\Theta}_{k-1} r_k^d \tag{26}$$

This control law is for discussion purposes only and will be modified subsequently.

Let the *output tracking error* be defined as,

$$\mathcal{E}_k = Y_k - Y_k^d \tag{27}$$

Using adaptive control law (26) and the MP-RLS estimator, the output tracking error is related to the input prediction error as follows,

$$\tilde{E}_k = \frac{U_k^p - U_k}{1 + \eta_k} = \hat{\Theta}_{k-1}\tilde{r}_k - \hat{\Theta}_{k-1}\tilde{r}_k^d \tag{28}$$

$$= \frac{\hat{L}_{k-1}(Y_k - Y_k^d)}{1 + \eta_k} = \hat{L}_{k-1}\hat{\mathcal{E}}_k \tag{29}$$

where the normalized tracking error is defined as

$$\tilde{\mathcal{E}}_k = \frac{\mathcal{E}_k}{1+\eta_k} \tag{30}$$

Remark 1 For control purposes, it is desired for the output tracking error to converge to zero. Given that $\tilde{E}_k$ goes to zero by property P4 of the estimator, it is clear from (29) that $\mathcal{E}_k$ will also go to zero if $\underline{\sigma}(\hat{L}_{k-1})$ is bounded away from zero. Unfortunately, while the true gain L satisfies this property, the estimate L_k produced from the recursive estimation scheme has no such guaranteed properties. The possible singularity of the estimate $\hat{L}_k$ destroys the above argument for convergence of the tracking error and is the essence of the difficulty associated with proving stability.

3.5 Adaptive Control Law - Modified

Lozano overcame the singularity problem for indirect adaptive control in [20] by introducing a modification of the matrix estimate baaed on a polar decomposition. A similar approach will be used here for direct adaptive control.

Construct the modified estimate,

$$\bar{\Theta}_k = \hat{\Theta}_k + \mu_k R_k F_k \tag{31}$$

$$R_k = [0 \,|Q_k] \tag{32}$$

where some lower bound is specified on μ,

$$\mu_k \geq \underline{\mu} > 0 \tag{33}$$

Here, matrix Q_k in (32) is determined from a *polar* decomposition,

$$\hat{L}_k = Q_k S_k \tag{34}$$

where Q_k is a real orthogonal matrix, and $S_k = S_k^T \geq 0$ (cf., Barnett [4]). Conceptually, the polar decomposition can be written in terms of the singular value decomposition $\hat{L}_k = U\Sigma V^T$ as follows,

$$\hat{L}_k = (UV^T)(V\Sigma V^T) \tag{35}$$

noting that $Q_k = UV^T$ is an orthogonal matrix and $S_k = V\Sigma V^T$ is symmetric non-negative definite by construction. The polar decomposition of a matrix gets its name from analogy to the polar decomposition of a complex number $z = e^{arg(z)}|z|$ since $S_k \geq 0$ plays the role of the nonnegative quantity $|z|$ and any unitary matrix Q can be written in the form e^{iW} with W Hermitian [4].-

Using the modified estimate (31), a modified adaptive control law can be defined aa,

Modified Adaptive Control

$$U_k = \bar{\Theta}_{k-1} r_k^d \tag{36}$$

$$\bar{\Theta}_{k-1} = [\bar{K}_{k-1} \,|\, \bar{L}_{k-1}] \tag{37}$$

where,

$$\bar{K}_{k-1} = \hat{K}_{k-1} + \mu_{k-1} Q_{k-1} f_{k-1}^T m_{k-1} \tag{38}$$

$$\bar{L}_{k-1} = \hat{L}_{k-1} + \mu_{k-1} Q_{k-1} f_{k-1}^T f_{k-1} \tag{39}$$

and m_{k-1} and f_{k-1} form the partitioned Cholesky factors of F_{k-1}, i.e.,

$$F_{k-1} = \mathcal{F}_{k-1}\mathcal{F}_{k-1}^T \geq 0 \tag{40}$$

$$\mathcal{F}_{k-1} = \begin{bmatrix} m_{k-1}^T \\ f_{k-1}^T \end{bmatrix} \tag{41}$$

This direct adaptive control law is depicted in Fig. 2.

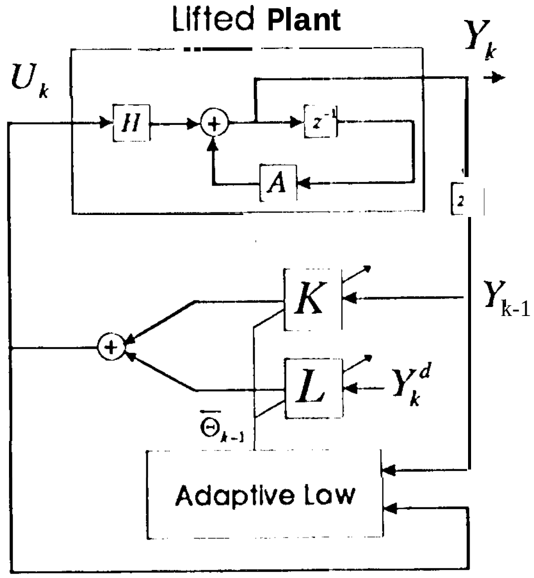

Figure 2: Stable direct adaptive periodic control law.

3.6 Stability Results

The main result is given next.

Theorem 1
Let the lifted plant (11) be controlled by the modified adaptive control (36) and MP-RLS estimation algorithm (24)(25), to follow a bounded trajectory $||Y_k^d|| \leq \kappa$. Then the signals U_k and Y_k remain bounded, and the tracking error goes to zero asymptotically, i.e.,

$$\lim_{k \to \infty} |Y_k - Y_k^d| = 0 \tag{42}$$

Proof: If the modified adaptive control (36) is applied to the plant (11) at each time k, the normalized input prediction error (20) becomes,

$$\tilde{E}_k = \frac{U_k^p - U_k}{1 + \eta_k} = \hat{\Theta}_{k-1}\tilde{r}_k - \overline{\Theta}_{k-1}\tilde{r}_k^d \tag{43}$$

$$= \hat{\Theta}_{k-1}\tilde{r}_k - \overline{\Theta}_{k-1}\tilde{r}_k^d \pm \mu_{k-1}R_{k-1}F_{k-1}\tilde{r}_k \tag{44}$$

$$= \tilde{\Theta}_{k-1}(\tilde{r}_k - \tilde{r}_k^d) - \mu_{k-1}R_{k-1}F_{k-1}\tilde{r}_k \tag{45}$$

$$= \frac{\tilde{L}_{k-1}(Y_k - Y_k^d)}{1 + \eta_k} - \mu_{k-1}R_{k-1}F_{k-1}\tilde{r}_k \tag{46}$$

Taking the limit of both sides of (46) and applying (P4) and (P8) yields,

$$\lim_{k\to\infty} \frac{\overline{L}_{k-1}(Y_k - Y_k^d)}{1+\eta_k} = 0 \tag{47}$$

Since by Lemma A2 of Appendix A, $\bar{\sigma}(\overline{L}_{k-1}) > 0$ is bounded away from zero, it follows from (47) that,

$$\lim_{k\to\infty} \frac{\mathcal{E}_k}{1+\eta_k} = 0 \tag{48}$$

Note also that,

$$\frac{\|\mathcal{E}_k\|^2}{(1+\eta_k)^2} \geq \frac{1}{2} \cdot \frac{\|\mathcal{E}_k\|^2}{1+\eta_k^2} \tag{49}$$

where we have used the fact that $2\eta_k \leq 1 + \eta_k^2$. Combining results (48) and (49) it follows that,

$$\lim_{k\to\infty} \frac{\|\mathcal{E}_k\|^2}{1+\eta_k^2} = 0 \tag{50}$$

Now consider convergence. of the unnormalized tracking error $\mathcal{E}_k$. Using the triangle inequality, one can verify the following linear bound edness condition,

$$\eta_k = \gamma_k(\|Y_{k-1}\| + \|Y_k\|) \leq \bar{\gamma}\|Y_{k-1} - Y_{k-1}^d\| + \bar{\gamma}\|Y_k - Y_k^d\| + \bar{\gamma}(\|Y_{k-1}^d\| + \|Y_k^d\|) \tag{51}$$

$$= \bar{\gamma}\|\mathcal{E}_{k-1}\| + \bar{\gamma}\|\mathcal{E}_k\| + c_1 \leq c_1 + c_2 \max_{0\leq\tau\leq k} \|\mathcal{E}_\tau\| \tag{52}$$

where $c_1 = 2\kappa\bar{\gamma} \geq \bar{\gamma}(\|Y_{k-1}^d\| + \|Y_k^d\|)$ and $c_2 = 2\bar{\gamma}$. Given convergence of (50) and linear boundedness rendition (52), the Key Technical Lemma (Goodwin and Sin [15]) ensures that,

$$\lim_{k\to\infty} \mathcal{E}_k = 0 \tag{53}$$

and that η_k remains bounded. Boundedness of η_k implies the boundedness of Y_k which together with P3, P5, and (36) imply the boundedness of U_k. ∎

Remark 2 In light of the discussion in Remark 1, the main idea behind the stability proof can be understood completely from (46), This relation uses the modified gain $\overline{L}$ and has the extra term $R_{k-1}F_{k-1}\tilde{r}_k$ compared with the error (29) which arises from using the unmodified gain $\hat{L}$. This extra term is due to the modification (31) of the parameter estimate. Somewhat remarkably, this term vanishes by property P8 of the estimator. Since the modified estimate $\overline{L}_{k-1}$ is nonsingular by design (i.e., Lemma A2), the stability proof outlined in Remark 1 is recovered.

It appears that property P8 was first used for proving adaptive stability in the paper by Lozarro and Goodwin [23], although the idea is implicit in an earlier paper by P. de Larminat [26]. In [23], P8 follows as a property of the normalized RLS algorithm with constant trace. Although the constant trace is dropped in the present MP-RLS algorithm, it is shown in Appendix B (ace also [7]), that property P8 is recovered by using data normalization in combination with convergent covariance propagation. ∎

4 NUMERICAL EXAMPLE

4.1 Two Cart Model

The direct adaptive control algorithm will be demonstrated on the two cart model shown in Fig. 3. The two carts have mass $m_1 = m_2 = 1$ and are connected with a spring having constant $k = 1$. It is desired to control the position x_2 of the second cart by applying a force u on the first cart, where the position x_2 is the measured variable. The transfer function in the Laplace Transform domain is given as,

$$\frac{x_2(s)}{u(s)} = \frac{k}{m_1 s^2 \left(m_2 s^2 + (1 + \frac{m_2}{m_1})k\right)} \tag{54}$$

A zero-hold discretization of the transfer function (54) with sampling time $T = 1$ gives the discrete-time system,

$$\frac{x_2(z)}{u(z)} = \frac{B(z)}{A(z)} \tag{55}$$

where the roots of B(z) are (-8.7103, , -1, -.1148) and the roots of $A(z)$ are (0.1559+ .9878i, 1, 1). It is seen that the plant haa unstable double integrator dynamics, and has nonminimum-phase zeros on and outside of the unit circle.

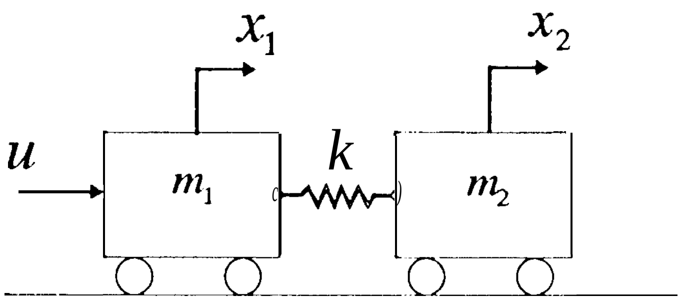

Figure 3: Two Cart Model

4.2 Example 1: Direct Adaptive Periodic Control

The adaptive estimator is initialized by $\Theta_0 = 0$, $F_0 = 10^{10}.1$, with design parameters $\gamma_k \equiv 10^{-20}$, $\mu_k \equiv .1/tr\{F_0\} = 1.25 \times 10^{-12}$. The carts are initialized with positions $x_1(0) = x_2(0) = 0$ and velocities $\dot{x}_1(0) = .1 \times 10^{-4}$, $\dot{x}_2(0) = .3 \times 10^{-4}$. The reference trajectory Y_k^d is chosen as a unit square wave with an 80 second period.

Simulation results are. shown in Figs. 4, 5, and 6. It is seen from Fig. 4 that the adaptively controlled system converges during the first 1.5 cycles of the square wave reference. Fig. 5 shows that the adaptive gains and covariance converge within the same period of time,

Even though the design parameter $\mu_k = 1.25 \times 10^{-12}$ has been chosen small in this example, it has a critical effect on the overall stability. In particular, the plot of $\underline{\sigma}(L_k)$ in Fig. 6 (bottom) shows that the matrix gain $\hat{L}_k$ is initially singular (less that 10^{-17} for double precision), and stays near-singular for at least 40 seconds. In contrast, the modified gain $\underline{\sigma}(\bar{L}_k)$ shown in Fig. 6 (top), remains bounded below for all time, as required for adaptive stability.

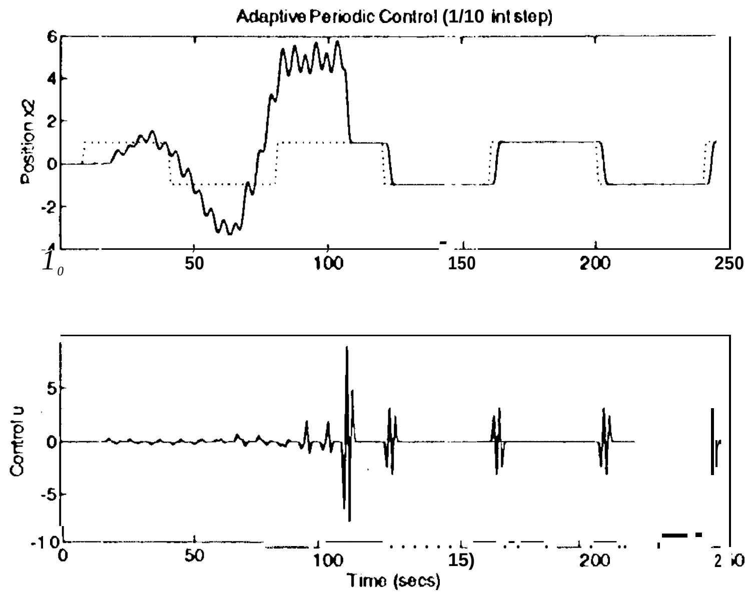

Figure 4: Direct adaptive control for two-cart model. *top:* position x_2 of second cart (solid), reference trajectory Y_k^d (dotted); *bottom:* control input U_k

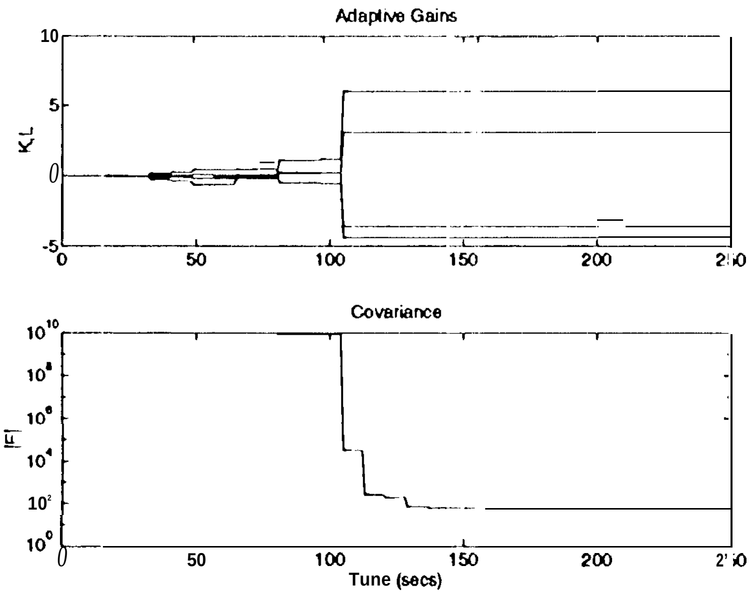

Figure 5: Direct adaptive control for two-cart model (cent'd). *top:* adaptive gains $\widehat{\Theta}_k$; *bottom:* covariance $\bar{\sigma}(F_k)$

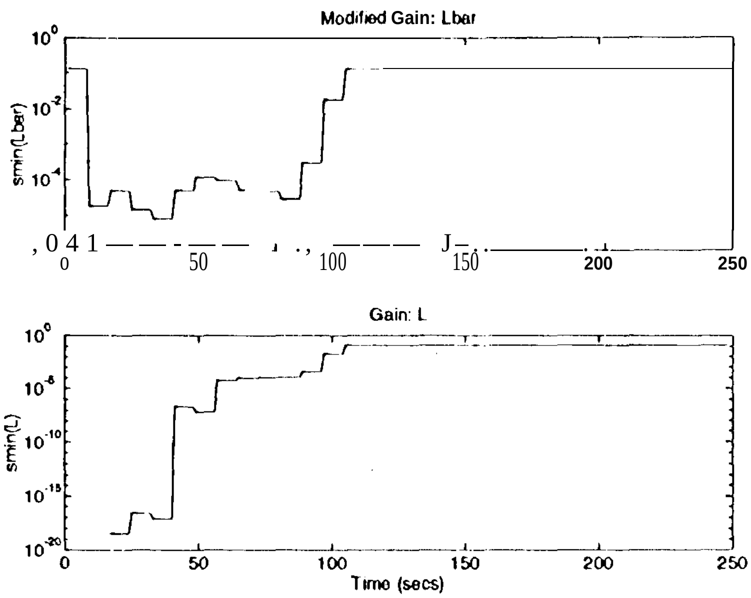

Figure 6: Direct adaptive control for two-cart model (cent'd). *top:* modified adaptive gain $\underline{\sigma}(\bar{L}_k)$; *bottom:* unmodified adaptive gain $\underline{\sigma}(\hat{L}_k)$

4.3 Example 2: Performance Considerations

Aside from choosing μ_k and γ_k positive for stability reasons, practical choices for these design parameters are driven by performance considerations. Typically these parameters should be chosen small so as to recover (in the limit), the nice transient response properties of the unmodified/unnormalized adaptive RLS algorithm shown by simulation in Jakubowski et. al. [16][17].

The effect of not choosing μ_k sufficiently small is shown by simulation and briefly analyzed. The set-up is identical to Example 1, except $\mu_k \equiv 1$. The results are shown in Fig. 7 where it is seen that the transient is on the order of 10^{42}.

The poor transient performance in this case can be traced to short periods of time during which the adaptive controller is "locally" unstable. Specifically, the time-varying closed-loop system can be calculated as,

$$Y_k = \mathcal{A}_k Y_{k-1} + H\bar{L}_{k-1} Y_k^d \tag{56}$$

where,

$$\mathcal{A}_k = A + H(\hat{K}_{k-1} + \mu_{k-1} Q_{k-1} f_{k-1}^T m_{k-1}) \tag{57}$$

Hence, for "local stability" the eigenvalues of the closed-loop system matrix $\mathcal{A}_k$ should be inside the unit circle. Since m_{k-1} and f_{k-1} are factors of the covariance F_{k-1} which is chosen large initially, the local stability condition will be violated unless μ_{k-1} in (57) is chosen *sufficiently small relative to the covariance.* Scaling μ to the reciprocal covariance trace (e.g., in the first simulation we chose $\mu_{k-1} \equiv .1/tr\{F_0\}$), is a reasonably good rule of thumb. While such choices are not required for stability, they are necessary (although not sufficient) conditions for a good transient response.

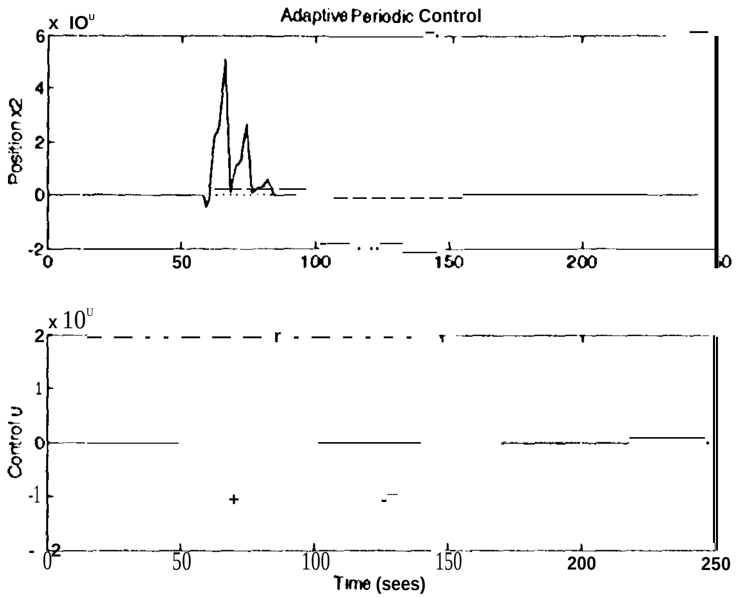

Figure 7: Simulation showing large transient due to choosing $\mu_k > 0$ too large. *top:* position x_2 of second cart (solid), reference trajectory Y_k^d (dotted); *bottom:* control input U_k

5 CONCLUSIONS

It is shown that only knowledge of the plant order is required to achieve stable direct adaptive control of nonminimum phase systems using periodic controllers. This relaxes requirements for stability found in earlier direct adaptive periodic control approaches involving plant Cauchy Index constraints [25], or partial plant Markov parameter knowledge [5].

As a result, stability requirements for convergence of direct adaptive periodic controllers are now on equal footing with requirements for indirect adaptive periodic control, as established in the work of Lozano [20].

Despite theoretical stability results, there are several open issues which remain to be resolved before the present approach can be made to work reliably in practice:

11. Reduction of adaptive transient

12. Modifications to meet actuator saturation constraints

13. Robustness to bounded process/measurement noise

14. Robustness to model order/delay, unmodelled dynamics

Concerning 11 and 12, large transients are often experienced when simulating systems with adaptive periodic controllers. This is partly due to the certainty *equivalence* property of the adaptive control which is controlling the *wrong plant with* conviction most of the time, In addition, even the transient response in the nonadaptive case can be large due to the fast "inverse plant" nature of the control. Unfortunately, pole-placement strategies offer little relief since poles of the lifted system are associated with the slow time scale and hence must be kept near the origin to maintain reasonable performance, For the nonadaptive case, it has been shown in [6] [8] that transients and control *signals can* be significantly reduced using extended horizon liftings. It is hoped that this same approach can lead to reduced transients in the adaptive case.

The algorithm in the present paper is not robust to bounded noise, and serves primarily to show equivalence of stability conditions between direct and indirect approaches under ideal conditions. Modifications similar to the deadzone in [20] are presently under consideration to address issue 13 in the direct adaptive case.

Issue 14 is perhaps the most difficult to address. The warnings contained in Goodwin and Feuer [14] regarding generalized sampling methods are most relevant for issue 14, since one must rely on high frequency plant dynamics for reliable control over low frequencies. A method proposed in Lozano [21] is applicable to overparametrization in the regulation problem, but presently has no extension to the tracking problem. Alternative approached based *on* multiple model banks are emerging, and may play an important role in the future (cf., Morse [24]).

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A APPENDIX A: Supporting Results

The whole point of Appendix A is show that the modified gain $\underline{\sigma}(\bar{L}_k)$ is bounded away from zero. Lemma Al is used to prove Lemma A2 which contains the desired result.

Lemma Al: Let the MP-RLS algorithm (24)(25) be applied to the normalized error equation (23). Then the estimate $\hat{L}_k$ and its polar factor S_k in (34) are explicitly bounded from above as follows,

$$\hat{L}_k \hat{L}_k^T \leq 2\alpha \cdot I \tag{A.1}$$

$$S_k \leq \sqrt{2\alpha} \cdot I \tag{A.2}$$

where,

$$\alpha \triangleq tr\{\Theta\Theta^T\} + v_0 \cdot \bar{\sigma}(F_0) \tag{A.3}$$

Proof: Consider the matrix inequality,

$$(X+Y)(X+Y)^T \leq 2(XX^T + YY^T) \tag{A.4}$$

Letting $X = \Theta$ and Y = Φ_k in (A.4) and using definition (19) gives,

$$\hat{\Theta}_k \hat{\Theta}_k^T \leq 2(@@+ \Phi_k \Phi_k^T) \tag{A.5}$$

At this point, one can construct the following sequence of inequalities,

$$\begin{aligned}
\hat{L}_k \hat{L}_k^T &\leq \hat{\Theta}_k \hat{\Theta}_k^T & \text{(A.6)}\\
&\leq 2(\Theta\Theta^T + \Phi_k \Phi_k^T) & \text{(A.7)}\\
&\leq 2\, tr\{\Theta\Theta^T + \Phi_k \Phi_k^T\} \cdot I & \text{(A.8)}\\
&\leq 2(\, tr\{\Theta\Theta^T\} + v_0 \cdot \bar{\sigma}(F_0)) \cdot I = 2\alpha \cdot I & \text{(A.9)}
\end{aligned}$$

Here, inequality (A.6) follows from the fact that $\hat{\Theta}_k \hat{\Theta}_k^T = \hat{K}_k \hat{K}_k^T + \hat{L}_k \hat{L}_k^T$; Inequality (A.7) follows from (A.5); Inequality (A.8) follows from the fact that $X \leq tr\{X\} \cdot I$ for any symmetric non-negative definite matrix X; Inequality (A.9) follows from property **P5** of the estimator; and definition of α in (A.3). This proves result, (A.1).

Using the polar decomposition (34) in (A.9) gives the relation,

$$Q_k S_k^2 Q_k^T \leq 2\alpha \cdot I \tag{A.10}$$

Hence, for any vector y,

$$y^T S_k^2 y = y^T Q_k^T (Q_k S_k^2 Q_k^T) Q_k y \leq 2\alpha \cdot y^T Q_k^T Q_k y = 2\alpha \, ||y||^2 \tag{A11}$$

where use has been made of (A.1O) and the orthogonality property $Q_k^T Q_k = I$. Since y is arbitrary in (All), one can conclude that,

$$S_k^2 \leq 2\alpha \cdot I \tag{A.12}$$

which gives (A.2) upon taking the square root. •

Lemma A2: Let b_0 be a positive scalar such that,

$$L^T L \geq b_0 \cdot I > 0 \tag{A.13}$$

Let the MP-RLS algorithm (24)(25) be applied to normalized error equation (23). Then the gain modification defined by (31) ensures that,

$$\bar{L}_k^T \bar{L}_k \geq \frac{b_0^2}{\rho^2} \cdot I > 0 \tag{A.14}$$

where,

$$\rho \triangleq 2 \cdot \max(\frac{v_0}{\underline{\mu}}, \sqrt{2\alpha}) \tag{A.15}$$

$$\alpha \triangleq tr\{\Theta\Theta^T\} + v_0 \cdot \bar{\sigma}(F_0) \tag{A.16}$$

Proof: Define,

$$\beta_k \triangleq \Phi_k \mathcal{F}_k^{-T} \tag{A.17}$$

Rearranging (A.17) and using (41) gives,

$$\Theta = \hat{\Theta}_k - \beta_k \mathcal{F}_k^T \tag{A.18}$$

$$L = \hat{L}_k - \beta_k f_k \tag{A.19}$$

Applying the matrix inequality,

$$(X - Y)^T(X - Y) \leq 2(X^T X + Y^T Y) \tag{A.20}$$

with choices X = $\hat{L}_k$ and Y = $\beta_k f_k$ to (A.19) give%,

$$L^T L \leq 2 \cdot \left(\hat{L}_k^T \hat{L}_k + f_k^T \beta_k^T \beta_k f_k\right) \tag{A.21}$$

At this point one can construct the following sequence of inequalities,

$$
\begin{aligned}
L^T L &\leq 2\left(\hat{L}_k^T \hat{L}_k + f_k^T f_k \cdot tr\{\beta_k^T \beta_k\}\right) && \text{(A.22)}\\
&\leq 2\left(\hat{L}_k^T \hat{L}_k + f_k^T f_k \cdot tr\{\Phi_k F_k^{-1} \Phi_k^T\}\right) && \text{(A.23)}\\
&\leq 2\left(\hat{L}_k^T \hat{L}_k + f_k^T f_k v_0\right) && \text{(A.24)}\\
&= 2\left(S_k^2 + f_k^T f_k v_0\right) && \text{(A.25)}\\
&\leq 2\left(\sqrt{2\alpha} \cdot S_k + f_k^T f_k v_0\right) && \text{(A.26)}\\
&\leq \rho\left(S_k + \mu_k f_k^T f_k\right) && \text{(A.27)}\\
&= \rho Q_k^T\left(Q_k S_k + \mu_k Q_k f_k^T f_k\right) = \rho Q_k^T \bar{L}_k && \text{(A.28)}
\end{aligned}
$$

Here, inequality (A.22) follows from (A.21) by using the matrix inequality $X^T Y X \leq X^T X \cdot tr\{Y\}$ valid for any symmetric non-negative definite Y; Inequality (A.23) follows by using the definition of β_k in (A. 17), Cholesky factors (40), and properties of the trace; Inequality (A .24) follows by property P2 of the estimator; Equality (A.25) follows by substituting the polar decomposition (34); Inequality (A.26) follows by result (A.2) of Lemma Al; Inequality (A.27) follows by the definition of ρ in (A.15); and equation (A.28) follows by the orthogonality of Q_k and the structure of the modified gain $\bar{L}_k$ in (39).

Using (A.13) and (A.28) gives upon squaring,

$$0 \leq b_0^2 \cdot I \leq (L^T L)^T (L^T L) \leq \rho^2 \bar{L}_k^T Q_k Q_k^T \bar{L}_k = \rho^2 \bar{L}_k^T \bar{L}_k \tag{A.29}$$

Rearranging, gives the desired result (A.14). •

B APPENDIX B: Normalized Matrix RLS Properties

To simplify the presentation, the notation and results in this appendix are self-contained

Consider measurements of the form,

$$Y_t = \Theta^o X_t; \; t = 1, ..., N \tag{B.1}$$

where $\Theta^o \in R^{m \times \ell}$ is an matrix of unknown parameters, and $Y_t \in R^m$, Xi $\in R^\ell$ are known measurement and regressor vectors, respectively. It is desired to recursively estimate the matrix Θ^o.

Lemma **B.1** Consider *the least squares cost* function,

$$\min_\Theta C_N = \min_\Theta \sum_{t=1}^{N} ||Y_t - \Theta X_t||^2 + tr\{(\Theta - \Theta_0)\bar{F}^{-1}(\Theta - \Theta_0)\} \tag{B.2}$$

where $\bar{F} = \bar{F}^T > 0$. Then the minimizing solution, *denoted as $\hat{\Theta}_N$, is* given *by iterating from $t = 1, ..., N$ on the following* recursive equations,

$$\hat{\Theta}_t = \hat{\Theta}_{t-1} + \frac{E_t X_t^T F_{t-1}}{1 + X_t^T F_{t-1} X_t} \tag{B.3}$$

$$F_t = \left(F_{t-1} - \frac{F_{t-1} X_t X_t^T F_{t-1}}{1 + X_t^T F_{t-1} X_t}\right); \; t = 1, ..., N \tag{B.4}$$

where $E_t = Y_t - \hat{\Theta}_{t-1} X_t$, $\hat{\Theta}_0 = \Theta_0$ and $F_0 = \bar{F}$.

Proof: This can be proved by separating the cost into scalar LS problems and applying standard results (cf., Bayard [7]), or by directly taking a matrix derivative of the cost (cf., Jakubowski [16]). ■

Theorem B.2 *Consider the measurement* equations (B.1) *where it is assumed that the regressor* is normalized as follows,

$$||X_t|| \leq \rho_x; \; t = 1, ..., \infty \tag{B.5}$$

Then the normalized matrix-parameter RLS algorithm (B.3)(B.4)(B.5), *has the* following properties,

P1 $\Phi_t F_t^{-1} = \Phi_{t-1} F_{t-1}^{-1} = ... = \Phi_0 F_0^{-1}$

P2-i $V_t = V_{t-1} - \frac{E_t E_t^T}{1 + X_t^T F_{t-1} X_t}$

P2-ii $v_t = v_{t-1} - \frac{||E_t||^2}{1 + X_t^T F_{t-1} X_t}$

P2-iii $v_t \leq v_{t-1} \leq \ldots \leq v_0$

P3 $\bar{\sigma}(F_t) \leq \bar{\sigma}(F_{t-1}) \leq \ldots \leq \bar{\sigma}(F_0)$

P4 $\lim_{t\to\infty} E_t = 0$

P5 $||\Phi_t||_f^2 \leq v_0 \cdot \bar{\sigma}(F_0)$

P6 $\lim_{t\to\infty} ||\hat{\Theta}_t - \hat{\Theta}_{t-1}||_f = 0$

P7 $\lim_{t\to\infty} F_t = F_\infty$

P8 $\lim_{t\to\infty} F_{t-1}X_t = 0$

P9 $\lim_{t\to\infty} \hat{\Theta}_t = \Theta_\infty = \Theta^\circ + \Phi_0 F_0^{-1} F_\infty$

where the Frobenious norm is defined as $||X||_f = (tr\{X^T X\})^{\frac{1}{2}}$ *and,*

$$\Phi_t \triangleq \hat{\Theta}_t - \Theta^\circ \tag{B.6}$$

$$v_t \triangleq tr\{V_t\}; \qquad V_t \triangleq \Phi_t F_t^{-1} \Phi_t^T \geq 0 \tag{B.7}$$

$$E_t \triangleq Y_t - \hat{\Theta}_{t-1} X_t = (\Theta^\circ - \hat{\Theta}_{t-1}) X_t = -\Phi_{t-1} X_t \tag{B.8}$$

Proof: The discussion here extends the results in Lozano-Leal and Goodwin [23] (Theorem 2.1, page 670-671), to the matrix parameter case.

Proof of PI: Multiplying (B.4) on the right by X_t and rearranging gives,

$$F_t X_t = F_{t-1} X_t - \frac{F_{t-1} X_t X_t^T F_{t-1} X_t}{1 + X_t^T F_{t-1} X_t} \tag{B.9}$$

$$= F_{t-1} X_t \left(I - \frac{X_t^T F_{t-1} X_t}{1 + X_t^T F_{t-1} X_t} \right) = \frac{F_{t-1} X_t}{1 + X_t^T F_{t-1} X_t} \tag{B.10}$$

Also from (B.4) and the matrix inversion Lemma [2],

$$F_t^{-1} = F_{t-1}^{-1} + X_t X_t^T \tag{B.11}$$

Using (B.8) in RLS update law (B.3) gives,

$$\hat{\Theta}_t = \hat{\Theta}_{t-1} - \frac{\Phi_{t-1} X_t X_t^T F_{t-1}}{1 + X_t^T F_{t-1} X_t} \tag{B.12}$$

Subtracting Θ° from both sides of (B.12) gives,

$$\Phi_t = \Phi_{t-1} - \frac{\Phi_{t-1} X_t X_t^T F_{t-1}}{1 + X_t^T F_{t-1} X_t} \tag{B.13}$$

$$= \Phi_{t-1} - \Phi_{t-1} X_t X_t^T F_t \tag{B.14}$$

$$= \Phi_{t-1}(I - X_t X_t^T F_t) \tag{B.15}$$

where (B. 14) follows from (B. 10), and (B. 15) follows by rearranging. Finally, using (B.11) in (B.15) gives,

$$\Phi_t = \Phi_{t-1}(I - F_t^{-1}F_t + F_{t-1}^{-1}F_t) \tag{B.16}$$

$$= \Phi_{t-1}F_{t-1}^{-1}F_t \tag{B.17}$$

Multiplying both sides of (B.17) on the right by F_t^{-1} gives PI, as desired.

Proof of P2: Multiplying each side of P1 on the right by the respective side of (B.13) (transposed) gives the identity,

$$\Phi_t F_t^{-1}\Phi_t^T = \Phi_{t-1}F_{t-1}^{-1}\left(\Phi_{t-1}^T - \frac{F_{t-1}X_t X_t^T \Phi_{t-1}^T}{1 + X_t^T F_{t-1}X_t}\right) \tag{B.18}$$

Using definition (B.7) in (B.18) and rearranging gives,

$$V_t = V_{t-1} - \frac{\Phi_{t-1}X_t X_t^T \Phi_{t-1}^T}{1 + X_t^T F_{t-1}X_t} \tag{B.19}$$

$$= V_{t-1} - \frac{E_t E_t^T}{1 + X_t^T F_{t-1}X_t} \tag{B.20}$$

which proves P2-i. Taking the trace of both sides of (B.20) and using (B.7) proves P2-ii. Property P2-iii follows directly from P2-ii.

Proof of P3: Taking the minimum singular value of both sides of (B.11) gives,

$$\underline{\sigma}(F_t^{-1}) = \underline{\sigma}(F_{t-1}^{-1} + X_t X_t^T) \geq \underline{\sigma}(F_{t-1}^{-1}) \tag{B.21}$$

or equivalently,

$$\frac{1}{\underline{\sigma}(F_t^{-1})} \leq \frac{1}{\underline{\sigma}(F_{t-1}^{-1})} \tag{B.22}$$

Property P3 follows from (B.22) and the fact that $\overline{\sigma}(X) = 1/\underline{\sigma}(X^{-1})$ for any nonsingular matrix X.

Proof of P4: Note that $v_\infty = \lim_{t\to\infty} v_t$ exists since from P2 the sequence v_t is monotonic nonincreasing and bounded below by O. Hence, rearranging P2 and summing both sides from 1 to ∞ gives,

$$v_0 - v_\infty = \sum_{t=1}^{\infty} \frac{E_t^T E_t}{1 + X_t^T F_{t-1}X_t} \tag{B.23}$$

$$\geq \sum_{t=1}^{\infty} \frac{E_t^T E_t}{1 + \overline{\sigma}(F_{t-1})\rho_x^2} \tag{B.24}$$

$$\geq \frac{1}{1 + \overline{\sigma}(F_0)\rho_x^2} \sum_{t=1}^{\infty} E_t^T E_t \tag{B.25}$$

where (B.25) follows from result P3. Result P4 is proved by noting that the nonegative sequence $E_t^T E_t$ in (B.25) is summable and hence approached zero asymptotically.

Proof of **P5:** From P2, it follows that $v_t \le v_0$. Using definition @.7) gives upon rearranging,

$$
\begin{aligned}
v_0 &\ge v_t && \text{(B.26)}\\
&= tr\{\Phi_t F_t^{-1}\Phi_t^T\} \ge \underline{\sigma}(F_t^{-1}) tr\{\Phi_t\Phi_t^T\} \\
&= \frac{tr\{\Phi_t\Phi_t^T\}}{\overline{\sigma}(F_t)} \ge \frac{tr\{\Phi_t\Phi_t^T\}}{\overline{\sigma}(F_0)} && \text{(B.27)}
\end{aligned}
$$

Crossmultiplying by $\overline{\sigma}(F_0)$ in @.27) and using the Frobenious norm definition gives result P5.

Proof of **P6:** From the RLS update @.3),

$$||\hat{\Theta}_t - \hat{\Theta}_{t-1}||_f^2 = tr\left\{\frac{E_t X_t^T F_{t-1} F_{t-1} X_t E_t^T}{(1 + X_t^T F_{t-1} X_t)^2}\right\} \tag{13.28}$$

$$\le \frac{\overline{\sigma}(F_{t-1}) X_t^T F_{t-1} X_t (E_t^T E_t)}{(1 + X_t^T F_{t-1} X_t)^2} \le \frac{\overline{\sigma}(F_{t-1}) E_t^T E_t}{1 + X_t^T F_{t-1} X_t} \tag{B.29}$$

$$\le \overline{\sigma}(F_{t-1}) E_t^T E_t \le \overline{\sigma}(F_0) E_t^T E_t \tag{B.30}$$

where (B.29) follows from the fact that $|x|/(1+|x|) \le 1$, and (B.30) follows from result P3. Using result P4 in (B.30) proves result **P6** as desired.

Proof of **P** 7: Before considering convergence of F_t, consider convergence of the symmetric *product* $z^T F_t z$ for any specified z. From the covariance update (B.4) it follows that,

$$z^T F_t z = z^T F_{t-1} z - r_{t-1} \ge O \tag{B.31}$$

where,

$$r_{t\ 1} \triangleq \frac{||z^T F_{t-1} X_t||^2}{1 + X_t^T F_{t-1} X_t} \ge 0 \tag{B.32}$$

Since $z^T F_t z$ *is* monotonic nonincreasing and bounded below by zero, it converges. Note that the ij'th entry f_{ij} of F_t *can* be always written as the asymmetric *product*,

$$f_{ij} = e_i^T F_t e_j \tag{B.33}$$

where e_i and e_j are unit vectors with 1's in the i'th and j'th elements, respectively. Convergence of f_{ij} follows by writing (B.33) as,

$$f_{ij} = e_i^T F_t e_j = \Big((e_i + e_j)^T F_t (e_i + e_j) - e_i^T F_t e_i - e_j^T F_t e_j\Big)/2 \tag{B.34}$$

and by invoking the previous convergence result for symmetric products. The matrix F_t is convergent since it is componentwise convergent.

Proof of **P8:** Taking the limit $t \to \infty$ of the trace of (B.4), and using result P7 gives,

$$\lim_{t\to\infty} \frac{||F_{t-1}X_t||^2}{1 + X_t^T F_{t-1} X_t} = 0 \tag{B.35}$$

However, from P3 it follows that,

$$\frac{||F_{t-1}X_t||^2}{1 + X_t^T F_{t-1} X_t} \geq \frac{||F_{t-1}X_t||^2}{1 + \bar{\sigma}(F_{t-1})\rho_x^2} > \frac{||F_{t-1}X_t||^2}{1 + \bar{\sigma}(F_0)\rho_x^2} \tag{B.36}$$

Taking the limit as $t + \infty$ in (B.36) and using (B.35) gives the desired result.

Proof of P9: Rearrange result PI to give, $\Phi_t = \Phi_0 F_0^{-1} F_t$, and take the limit $t \rightarrow \infty$ noting that $F_t \rightarrow F_\infty$ by result P7.

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